ISRAEL JOURNAL OF MATHEMATICS **166** (2008), 1–16
DOI: 10.1007/s11856-008-1017-y

RESOLVABILITY AND MONOTONE NORMALITY*

BY

István Juhász, Lajos Soukup

Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15
H-1053, Budapest, Hungary
e-mail: juhasz@renyi.hu, soukup@renyi.hu

AND

Zoltán Szentmiklóssy

Department of Analysis, Eötvös University of Budapest
Pázmány Péter Sétány 1/C,
H-1117 Budapest, Hungary
e-mail: zoli@renyi.hu

ABSTRACT

A space X is said to be κ-resolvable (resp., almost κ-resolvable) if it contains κ dense sets that are pairwise disjoint (resp., almost disjoint over the ideal of nowhere dense subsets). X is maximally resolvable if and only if it is $\Delta(X)$-resolvable, where $\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open}\}$.

We show that every crowded monotonically normal (in short: MN) space is ω-resolvable and almost μ-resolvable, where $\mu = \min\{2^\omega, \omega_2\}$. On the other hand, if κ is a measurable cardinal then there is a MN space X with $\Delta(X) = \kappa$ such that no subspace of X is ω_1-resolvable.

Any MN space of cardinality $< \aleph_\omega$ is maximally resolvable. But from a supercompact cardinal we obtain the consistency of the existence of a MN space X with $|X| = \Delta(X) = \aleph_\omega$ such that no subspace of X is ω_2-resolvable.

---

* The preparation of this paper was supported by OTKA grant no. 61600
Received August 25, 2006

1. ω-resolvability

For a topological space X we denote by $\mathcal{D}(X)$ the family of all dense subsets of X and by $\mathcal{N}(X)$ the ideal of all nowhere dense sets in X. Given a cardinal $\kappa > 1$, the space X is called **κ-resolvable** if and only if it contains κ many disjoint dense subsets. We say that X is **almost κ-resolvable** if there are κ many dense sets whose pairwise intersections are nowhere dense, that is, we have $\{D_\alpha : \alpha < \kappa\} \subset \mathcal{D}(X)$such that $D_\alpha \cap D_\beta \in \mathcal{N}(X)$ if $\alpha \neq \beta$. X is maximally resolvable if and only if it is $\Delta(X)$-resolvable, where $\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open}\}$ is called the dispersion character of X. Finally, if X is *not* κ-resolvable then it is also called **κ- irresolvable**.

The following simple but useful fact, in the case of κ-resolvability, was observed by El'kin in [5].

LEMMA 1.1: *A space X is κ-resolvable (almost κ-resolvable) if and only if every nonempty open set in X includes a nonempty (and open) κ-resolvable (almost κ-resolvable) subset.*

The aim of this paper is to investigate the (almost) resolvability properties of monotonically normal spaces. Since the most important examples of monotonically normal spaces are metric and linearly ordered spaces that are all known to be maximally resolvable, this aim seems to be both natural and justified to us. We hope that our results, by turning out to be both surprising and nontrivial, will also convince the reader about this.

Let us next recall the definition of monotonically normal spaces. For any topological space X we write

$$\mathcal{M}(X) = \big\{\langle x, U\rangle \in X \times \tau(X) : x \in U\big\}.$$

The elements of $\mathcal{M}(X)$ will be referred to as **marked** open sets. The space X is called **monotonically normal** if and only if it is T_1 and it admits a **monotone normality operator**, that is a function $H : \mathcal{M}(X) \to \tau(X)$ such that

(1) $x \in \mathrm{H}(x, U) \subset U$ for each $\langle x, U\rangle \in \mathcal{M}(X)$,
(2) if $\mathrm{H}(x, U) \cap \mathrm{H}(y, V) \neq \emptyset$ then $x \in V$ or $y \in U$.

We call a set D in a space X **strongly discrete** if the points in D may be separated by pairwise disjoint neighborhoods. It is well-known that in a monotonically normal space any discrete subset is strongly discrete. On the other hand, in [3] it was proved that every non-isolated point of a monotonically normal space is the accumulation point of a discrete subspace. Consequently, one obtains the following result.

THEOREM 1.2 ([3]): *In a monotonically normal space any non-isolated point is the accumulation point of some strongly discrete set.*

Let us say that a space X is SD if it has the property described in Theorem 1.2, that is every non-isolated point of X is the accumulation point of some strongly discrete set.

THEOREM 1.3: *Any crowded SD space X is ω-resolvable.*

Proof. The SD property is clearly hereditary for open subspaces. Hence, by Lemma 1.1, it suffices to prove that X includes an ω-resolvable subspace.

First we show that for every strongly discrete $D \subset X$ there is a strongly discrete $E \subset X \setminus \overline{D}$ such that $D \subset \overline{E}$. Indeed, fix a neighbourhood assignment U_d on D that separates D and for each $d \in D$ pick a strongly discrete set $E_d \subset X \setminus \{d\}$ with $d \in \overline{E_d}$. Then $E = \bigcup_{d \in D}(E_d \cap U_d)$ is clearly as claimed.

Now pick an arbitrary point $x \in X$ and set $E_0 = \{x\}$. Using the above claim, for each $n < \omega$ we can inductively define a strongly discrete set $E_{n+1} \subset X \setminus \overline{E_n}$ such that $E_n \subset \overline{E_{n+1}}$. Since $\bigcup_{i \le n} E_i \subset \overline{E_n}$, the sets $\{E_n : n < \omega\}$ are pairwise disjoint. Let us finally set $E = \bigcup\{E_n : n < \omega\}$. It is clear from our construction that if $I \subset \omega$ is infinite then $\bigcup\{E_n : n \in I\}$ is dense in E, so the subspace E of X is obviously ω-resolvable. ∎

COROLLARY 1.4: *Every crowded monotonically normal space is ω-resolvable.*

2. H-sequences and almost resolvability

The main result of the previous section, namely that (crowded) monotonically normal spaces are ω-resolvable, used very little of the particular structure provided by monotone normality. In this section we shall describe a procedure on monotonically normal spaces that is quite specific in this respect and so, not surprisingly, it leads to some stronger (almost) resolvability results. This procedure had been originated (in a different form) by S. Williams and H. Zhou in [13].

Definition 2.1: Let H be a monotone normality operator on a space X. A family $\mathcal{E} \subset \mathcal{M}(X)$ of marked open sets is said to be **H- disjoint** if for any two members

$\langle x, U\rangle, \langle y, V\rangle$ of $\mathcal{E}$ we have $\mathrm{H}(x, U) \cap \mathrm{H}(y, V) = \emptyset$. Clearly, if $\mathcal{E}$ is H-disjoint then $D(\mathcal{E}) = \{x : \exists U \text{ with } \langle x, U\rangle \in \mathcal{E}\}$ is (strongly) discrete.

By Zorn's lemma, for every open set G in X we can fix a maximal H-disjoint family $\mathcal{E}(G) \subset \mathcal{M}(G)$ with the additional property that $\overline{U} \subset G$ whenever $\langle x, U\rangle \in \mathcal{E}(G)$. The maximality of $\mathcal{E}(G)$ implies that

$$\bigcup\{\mathrm{H}(x, U) : \langle x, U\rangle \in \mathcal{E}(G)\}$$

is a dense open subset of G.

With the help of the above defined operator $\mathcal{E}(G)$ we may now describe our basic procedure as follows.

Definition 2.2: A sequence $\langle \mathcal{E}_\alpha : \alpha < \delta\rangle$ is called a completed H-sequence of X if and only if

(1) $\mathcal{E}_0 = \mathcal{E}(X)$;

(2) for each $\alpha < \delta$ we have

$$\mathcal{E}_{\alpha+1} = \bigcup \big\{\mathcal{E}\big(\,\mathrm{H}(x, U)\setminus\{x\}\big) : \langle x, U\rangle \in \mathcal{E}_\alpha\big\};$$

(3) if $\alpha < \delta$ is a limit ordinal then the family

$$\mathcal{W}_\alpha = \{W \in \tau(X) : \forall\beta < \alpha\ \exists\langle x, U\rangle \in \mathcal{E}_\beta \text{ with } W \subset U\}$$

is a π-base in X (or, equivalently, its union $\cup\mathcal{W}_\alpha$ is dense in X) and $\mathcal{E}_\alpha$ is a maximal H-disjoint collection of marked open sets $\langle y, V\rangle$ with $V \in \mathcal{W}_\alpha$;

(4) the family

$$\mathcal{W}_\delta = \{W \in \tau(X) : \forall\beta < \delta\ \exists\langle x, U\rangle \in \mathcal{E}_\beta \text{ with } W \subset U\}$$

is *not* a π-base in X.

The reader may convince himself by a straight-forward transfinite induction that the following fact is valid.

FACT 2.3: *Every crowded monotonically normal space X, with monotone normality operator* H, *admits a completed* H*-sequence $\langle \mathcal{E}_\alpha : \alpha < \delta\rangle$ where δ is necessarily a limit ordinal.*

We introduce now some notation concerning a given completed H-sequence $\langle \mathcal{E}_\alpha : \alpha < \delta\rangle$ of X. For any ordinal $\alpha < \delta$ we put $D_\alpha = D(\mathcal{E}_\alpha)$ and $H_\alpha = \bigcup\{\mathrm{H}(x, U) : \langle x, U\rangle \in \mathcal{E}_\alpha\}$. It is clear from our definitions that each H_α is dense open in X, moreover, $\beta < \alpha < \delta$ implies that $H_\beta \supset H_\alpha$ and $D_\beta \cap H_\alpha = \emptyset$. If $I \subset \delta$ is a set of ordinals we write $D[I] = \bigcup\{D_\alpha : \alpha \in I\}$. Finally, we set $V = X \setminus \overline{\cup\mathcal{W}_\delta}$, then V is a nonempty open set in X.

Lemma 2.4: *If I is bounded in δ then $D[I]$ is nowhere dense in X. However, if I is unbounded in δ then $D[I]$ is dense in V, that is we have $V \subset \overline{D[I]}$.*

Proof. The first part is obvious because $I \subset \alpha < \delta$ implies $D[I] \cap H_\alpha = \emptyset$.

Assume now that I is cofinal in δ but, arguing indirectly, for some $G \in \tau^*(V)$ we have $G \cap D[I] = \emptyset$. Pick any point $z \in G$, we claim that then, for all $\alpha < \delta$ and $\langle x, U\rangle \in \mathcal{E}_\alpha$, $\mathrm{H}(x,U) \cap \mathrm{H}(z,G) \neq \emptyset$ implies $z \in \mathrm{H}(x,U)$.

Indeed, if $\beta \in (\alpha, \delta) \cap I$ then there is $\langle x', U'\rangle \in \mathcal{E}_\beta$ with

$$\mathrm{H}(x', U') \cap \mathrm{H}(x,U) \cap \mathrm{H}(z,G) \neq \emptyset,$$

because H_β is dense in X. It follows that $U' \subset \mathrm{H}(x,U)$, hence $x' \notin G$ as $x' \in D_\beta$ and $G \cap D_\beta = \emptyset$. But then $\mathrm{H}(x', U') \cap \mathrm{H}(z,G) \neq \emptyset$ implies $z \in U' \subset \mathrm{H}(x,U)$.

The sets $\{\mathrm{H}(x,U) : \langle x,U\rangle \in \mathcal{E}_\alpha\}$ being pairwise disjoint, it follows that for each $\alpha < \delta$ there is exactly one $\langle x_\alpha, U_\alpha\rangle \in \mathcal{E}_\alpha$ such that $\mathrm{H}(x_\alpha, U_\alpha) \cap \mathrm{H}(z,G) \neq \emptyset$. But then $\mathrm{H}(z,G) \subset \overline{\mathrm{H}(x_\alpha, U_\alpha)} \subset \overline{U_\alpha}$ whenever $\alpha < \delta$, consequently

$$\mathrm{H}(z,G) \subset \overline{U_{\alpha+1}} \subset U_\alpha$$

as well. This, however, would imply $\mathrm{H}(z,G) \in \mathcal{W}_\delta$, contradicting that $\mathrm{H}(z,G) \subset G \subset V$. ∎

We may now give the main result of this section.

Theorem 2.5: *Any crowded monotonically normal space X is almost $\min(\mathfrak{c}, \omega_2)$-resolvable. So X is almost ω_1-resolvable, and even almost ω_2-resolvable if the continuum hypothesis (CH) fails.*

Proof. By Lemma 1.1 it suffices to show that some nonempty open $V \subset X$ satisfies this property. To see this, let us consider a completed H-sequence $\langle \mathcal{E}_\alpha : \alpha < \delta\rangle$ of X. Let I be a cofinal subset of δ of order type $\mathrm{cf}(\delta)$ and $\{I_\zeta : \zeta < \mu\}$ be an almost disjoint subfamily of $[I]^{\mathrm{cf}(\delta)}$, where $\mu = 2^\omega = \mathfrak{c}$ if $\mathrm{cf}(\delta) = \omega$ and $\mu = \mathrm{cf}(\delta)^+ \geq \omega_2$ if $\mathrm{cf}(\delta) > \omega$. Then the family $\{D[I_\zeta] : \zeta < \mu\}$ witnesses that V is almost μ-resolvable. ∎

Since almost ω-resolvability is clearly equivalent to ω-resolvability, Theorem 2.5 provides us a new proof of Corollary 1.4.

3. Spaces from trees and ultrafilters

Since the prime examples of monotonically normal spaces, namely metric and ordered spaces, are all maximally resolvable, the results of the two preceding sections seem rather modest. The main aim of this section is to show that, at least, modulo some large cardinals, nothing stronger than ω-resolvability can be expected of a monotonically normal space X, even if its dispersion character $\Delta(X)$ is large. The examples that show this have actually been around but, as far as we know, the fact that they are monotonically normal has not been noticed.

The underlying set of such a space is an **everywhere infinitely branching** tree $\langle T, < \rangle$. This simply means that for each $t \in T$ the set S_t of all immediate successors of t in T is infinite. The height of such a tree is obviously a limit ordinal. (In fact, nothing is lost if we only consider trees of height ω.) By a **filtration** on T we mean a map F with domain T that assigns to every $t \in T$ a filter $F(t)$ on S_t such that every cofinite subset of S_t belongs to $F(t)$ (that is, $F(t)$ extends the Fréchet filter on S_t).

Definition 3.1: Assume that F is a filtration on an everywhere infinitely branching tree $\langle T, < \rangle$. A topology τ_F is then defined on T by

$$\tau_F = \{V \subset T : \forall t \in V \ \big(V \cap S_t \in F(t)\big)\},$$

and the space $\langle T, \tau_F \rangle$ is denoted by $X(F)$.

THEOREM 3.2: *Let F be a filtration on an everywhere infinitely branching tree $\langle T, < \rangle$. Then the space $X(F)$ is monotonically normal.*

Proof. That τ_F is indeed a topology that satisfies the T_1 separation axiom is obvious and well-known. The novelty is in showing that $X(F)$ is monotonically normal.

To this end we define $\mathrm{H}(s, V)$ for $s \in V \in \tau_F$ as follows:

$$\mathrm{H}(s, V) = \{t \in V : s \le t \text{ and } [s, t] \subset V\}.$$

Of course, here $[s, t] = \{r : s \le r \le t\}$. Clearly, $\mathrm{H}(s, V) \in \tau_F$ and $s \in \mathrm{H}(s, V) \subset V$.

Next, assume that $t \in \mathrm{H}(s_1, V_1) \cap \mathrm{H}(s_2, V_2)$. Then $s_1, s_2 \le t$ implies that s_1 and s_2 are comparable, say $s_1 \le s_2$. But then we have $s_2 \in [s_1, t] \subset V_1$, consequently H is indeed a monotone normality operator on $X(F)$. ∎

Of special interest are those filtrations F for which $F(t)$ is a (free) ultrafilter on S_t for all $t \in T$. Such an F will be called an **ultrafiltration**. In this case we have a convenient way to determine the closures of sets in the space $X(F)$ that will be put to good use later.

Definition 3.3: For every set $A \subset T$ we define

$$C(A) = A \cup \{t \in T : S_t \cap A \in F(t)\}.$$

Then by transfinite recursion we define $C^\alpha(A)$ for all ordinals α by $C^{\alpha+1}(A) = C(C^\alpha(A))$ for successors and $C^\alpha(A) = \bigcup\{C^\beta : \beta < \alpha\}$ for α limit.

LEMMA 3.4: *Let F be an ultrafiltration on the tree T. Then a set $B \subset T$ is closed in $X(F)$ if and only if $B = C(B)$. Consequently, for any subset $A \subset T$ there is an ordinal $\alpha < |T|^+$ with $\overline{A} = C^\alpha(A)$.*

Proof. First, if $B = C(B)$ then for each $t \in T \setminus B$ we have $S_t \cap B \notin F(t)$, hence $S_t \setminus B \in F(t)$ because $F(t)$ is an ultrafilter. Then $T \setminus B$ is open by the definition of τ_F, hence B is closed. Conversely, if B is closed in $X(F)$ then for each $t \in T \setminus B$ we have $S_t \setminus B \in F(t)$, hence $S_t \cap B \notin F(t)$, that is $t \notin C(B)$. But this means that $B = C(B)$.

Next, $C(A) \subset \overline{A}$ is obvious, and then by induction we get $C^\alpha(A) \subset \overline{A}$ for all α. But for some $\alpha < |T|^+$ we must have $C(C^\alpha(A)) = C^\alpha(A)$, and then $\overline{A} = C^\alpha(A)$ for $C^\alpha(A)$ is closed by the above. ∎

Let $\mathfrak{u}$ be an ultrafilter on a set I and λ be a cardinal. $\mathfrak{u}$ is said to be **$\boldsymbol{\lambda}$-descendingly complete** if and only if $\bigcap\{X_\xi : \xi < \lambda\} \in \mathfrak{u}$ for each decreasing sequence $\{X_\xi : \xi < \lambda\} \subset \mathfrak{u}$. The ultrafilter $\mathfrak{u}$ is called **$\boldsymbol{\lambda}$-descendingly incomplete** if and only if it is not λ-descendingly complete. For example, $\mathfrak{u}$ is countably complete exactly if it is ω-descendingly complete.

We shall need the following old result of Kunen and Prikry in our next irresolvability theorem for spaces obtained from certain ultrafiltrations.

THEOREM (Kunen, Prikry, [10]): *If λ is a regular cardinal and $\mathfrak{u}$ is a λ-descendingly complete ultrafilter (on any set) then $\mathfrak{u}$ is also λ^+-descendingly complete.*

THEOREM 3.5: *Assume that F is an ultrafiltration on T and λ is a regular cardinal such that $F(t)$ is λ-descendingly complete for all $t \in T$. Then the space $X(F)$ is hereditarily λ^+-irresolvable (that is, no subspace of $X(F)$ is λ^+-resolvable).*

Proof. First we show that for every set $A \subset T$ we have $\overline{A} = C^\lambda(A)$. By Lemma 3.4 it suffices to show that $C(C^\lambda(A)) = C^\lambda(A)$.

Indirectly assume that $t \in C(C^\lambda(A)) \setminus C^\lambda(A)$, then we must have $C^\lambda(A) \cap S_t \in F(t)$. But

$$C^\lambda(A) \cap S_t = \bigcup_{\alpha<\lambda} C^\alpha(A) \cap S_t$$

where the right-hand side is an increasing union, hence there is an $\alpha < \lambda$ with $C^\alpha(A) \cap S_t \in F(t)$ because $F(t)$ is λ-descendingly complete. This implies that $t \in C^{\alpha+1}(A) \subset C^\lambda(A)$, a contradiction.

Let us now consider an *indexed* family of sets $\mathcal{F} = \{F_i : i \in I\}$. We are going to use the following notation:

$$\operatorname{ord}(x, \mathcal{F}) = |\{i \in I : x \in F_i\}|$$

and

$$\operatorname{ord}(\mathcal{F}) = \sup\{\operatorname{ord}(x, \mathcal{F}) : x \in \cup_{i \in I} F_i\}.$$

Instead of the statement of the theorem we shall prove the following much stronger claim.

LEMMA 3.6: *If $\mathcal{D} = \{D_i : i \in I\}$ is any indexed family of subsets of T with $\operatorname{ord}(\mathcal{D}) \le \lambda$ then $\operatorname{ord}(\{\overline{D_i} : i \in I\}) \le \lambda$ as well.*

Proof. We shall prove, by induction on $\alpha \le \lambda$, that $\operatorname{ord}(\mathcal{D}^\alpha) \le \lambda$ where

$$\mathcal{D}^\alpha = \{C^\alpha(D_i) : i \in I\}.$$

First we show that $\operatorname{ord}(\mathcal{D}^1) \le \lambda$, this will clearly take care of all the successor steps.

Assume, indirectly, that $\operatorname{ord}(t, \mathcal{D}^1) \ge \lambda^+$ for some $t \in T$, then we may find a set $J \in [I]^{\lambda^+}$ such that $t \in C(D_j) \setminus D_j$, hence $D_j \cap S_t \in F(t)$, for each $j \in J$.

By the theorem of Kunen and Prikry the ultrafilter $F(t)$ is also λ^+-descendingly complete. Consequently, using a standard argument, one can show that there is an $L \in \left[J\right]^{\lambda^+}$ such that

$$\bigcap\{D_j \cap S_t : j \in L\} \ne \emptyset.$$

But this clearly contradicts $\operatorname{ord}(\mathcal{D}) \le \lambda$.

Next assume that $\alpha \le \lambda$ is a limit ordinal and the inductive hypothesis holds for all $\beta < \alpha$. But now for each index $i \in I$ we have $C^\alpha(D_i) = \bigcup_{\beta<\alpha} C^\beta(D_i)$, hence

$$\mathrm{ord}(t, \mathcal{D}^\alpha) \leq \sum_{\beta<\alpha} \mathrm{ord}(t, \mathcal{D}^\beta) \leq |\alpha| \cdot \lambda = \lambda$$

whenever $t \in T$, and so $\mathrm{ord}(\mathcal{D}^\alpha) \leq \lambda$. ∎

It follows immediately from Lemma 3.6 that if $\{A_i : i \in \lambda^+\}$ are pairwise disjoint nonempty subsets of T then the closures $\overline{A_i}$ cannot all be the same and so no subspace of $X(F)$ can be λ^+-resolvable. ∎

COROLLARY 3.7: *If F is an ultrafiltration on T such that $F(t)$ is countably complete for each $t \in T$ then $X(F)$ is ω-resolvable but hereditarily ω_1-irresolvable. In particular, if κ is a measurable cardinal then there is a monotonically normal space X with $|X| = \Delta(X) = \kappa$ that is hereditarily ω_1-irresolvable.*

The question if ω-resolvable spaces are also maximally resolvable was raised a long time ago by Ceder and Pearson in [2], and has just recently been settled completely in [8] (negatively). Corollary 3.7 yields a monotonically normal counterexample to this problem, from a measurable cardinal. Another counterexample from a measurable cardinal was found by Eckertson in [4], however, that example is not monotonically normal. We present two arguments to show this. First, Eckertson's example contains a crowded irresolvable subspace, hence it cannot be monotonically normal by corollary 1.4.

The second argument is based on our following observation that may have some independent interest. First we need some notation. If $\kappa \leq \lambda$ are cardinals we let τ_κ^λ denote the $<\kappa$ box product topology on 2^λ (generated by the base $\{[f] : f \in Fn(\lambda, 2; \kappa\}$, where $[f] = \{x \in 2^\lambda : f \subset x\}$), moreover we set $\mathbb{C}_{\lambda,\kappa} = \left\langle 2^\lambda, \tau_\kappa^\lambda \right\rangle$.

THEOREM 3.8: *If $\kappa^{<\kappa} = \kappa < \lambda$ then no dense subspace of $\mathbb{C}_{\lambda,\kappa}$ is monotonically normal.*

Proof of 3.8. Let X be dense in $\mathbb{C}_{\lambda,\kappa}$ and θ be a large enough regular cardinal. Let $\mathcal{M}$ be an elementary submodel of $\langle \mathcal{H}(\theta), \in, \prec \rangle$ (where $\mathcal{H}(\theta)$ is the family of sets hereditarily of size $<\theta$ and $\prec$ is a well-ordering of $\mathcal{H}(\theta)$) such that $|\mathcal{M}| = \kappa$ and $\left[\mathcal{M}\right]^{<\kappa} \subset \mathcal{M}$, moreover $X, \kappa, \lambda \in \mathcal{M}$. Note that then $Fn(\left[\mathcal{M} \cap \lambda\right]^{<\kappa}, 2; \kappa) \subset \mathcal{M}$ as well.

Assume that X is monotonically normal and let $\mathrm{H} \in \mathcal{M}$ be a monotone normality operator on X. We can assume that $\mathrm{H}(x, [s] \cap X)$ is the trace on X of a basic open set for each basic open set $[s]$.

Let $I = \mathcal{M} \cap \lambda$ and pick $\alpha \in \lambda \setminus I$. $\mathcal{F} = \{f \restriction I : f \in \mathcal{M} \cap X\}$ is clearly dense in the subspace 2^I of $\mathbb{C}_{\lambda,\kappa}$. Let $\mathcal{F}_i = \{f \restriction I : f \in X \cap \mathcal{M} \wedge f(\alpha) = i\}$ for $i \in 2$ then $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ so there is $i \in 2$ and $s \in Fn(I, 2; \kappa)$ such that $\mathcal{F}_i$ is dense in $2^I \cap [s] \cap X$.

Let $b = s \cup \{\langle \alpha, 1 - i \rangle\}$ and pick $x \in X \cap [b]$. Next, let $\mathrm{H}(x, [b] \cap X) = [b'] \cap X$ and $b'' = b' \restriction I$. Fix $b''' \in Fn(I, 2; \kappa)$ such that $b''' \supset b''$ and $x \notin [b''']$. Since $\mathcal{F}_i$ is dense in $2^I \cap [s] \cap X$ we can pick $y \in X \cap \mathcal{M} \cap [b''']$ such that $y(\alpha) = i$. Let $[u] \cap X = \mathrm{H}(y, [b'''] \cap X)$. Then $\operatorname{dom} u \subset I$ because $\mathrm{H}, b''', y \in \mathcal{M}$.

Since $x \notin [b''']$ and $y \notin [b]$ it follows that $\mathrm{H}(x, [b]) \cap \mathrm{H}(y, [b''']) = [u] \cap [b'] \cap X = \emptyset$. However $\operatorname{supp} u \subset I$ and $u \supset b''' \supset b'' = b' \restriction I$, so u and b' are compatible functions of size $< \kappa$, i.e., $[u] \cap [b']$ is a nonempty open set in $\langle 2^\lambda, \tau_\kappa^\lambda \rangle$. Since X is dense we have $[u] \cap [b'] \cap X \neq \emptyset$, a contradiction. ∎

Now, Eckertson's example obtained from a measurable cardinal κ contains a subspace homeomorphic to a dense subspace of $\mathbb{C}_{2^\kappa,\kappa}$, hence it cannot be monotonically normal by Theorem 3.8 because $\kappa^{<\kappa} = \kappa$.

Of course, we have a space like in Corollary 3.7 if and only if there is a measurable cardinal. Also, the cardinality (and dispersion character) of such a space is at least as large as the first measurable. But can we have an example of a monotonically normal and not maximally resolvable space that is much smaller? The answer to this question is, consistently, affirmative but, ironically, it requires the existence of a large cardinal that is much stronger than a measurable.

Theorem (Magidor, [11]): *It is consistent from a supercompact cardinal that there is an ω_1-descendingly complete uniform ultrafilter on $\aleph_\omega$.*

We would like to emphasize that in [1] a slightly weaker result was given in which $\aleph_\omega$ is replaced with $\aleph_{\omega+1}$. However, Magidor pointed out to us that the method of [1] yields the above stronger version as well. From Magidor's theorem and from Theorem 3.5 we immediately obtain our promised result.

Corollary 3.9: *From a supercompact cardinal it is consistent to have a monotonically normal space X with $|X| = \Delta(X) = \aleph_\omega$ that is hereditarily ω_2-irresolvable (hence not maximally resolvable).*

Of course, from [1] we could conclude the slightly weaker result in which $\aleph_\omega$ is replaced with $\aleph_{\omega+1}$.

But can we do even better and go below $\aleph_\omega$? The answer to this question is, maybe surprisingly, negative. We are going to show that any monotonically normal space of cardinality less than $\aleph_\omega$ is maximally resolvable. The proof of this result will be based on showing that all spaces of the form $X(F)$ with F an ultrafiltration on the tree $\mathrm{Seq}\,\kappa = \kappa^{<\omega}$ of all finite sequences of ordinals less than κ are maximally resolvable provided that $\kappa < \aleph_\omega$. The first result to this effect, for constant ultrafiltrations on $\mathrm{Seq}\,\omega_n$, was obtained by László Hegedüs in his Master's Thesis [6]. Of course, by a constant ultrafiltration we mean one for which $F(t)$ is the "same" ultrafilter for all $t \in T$.

Now, let κ be an arbitrary infinite cardinal. A nonempty subset T of $\mathrm{Seq}\,\kappa$ is called a **subtree** of $\mathrm{Seq}\,\kappa$ if and only if $t \restriction n \in T$ whenever $t \in T$ and $n < |t|$. For any subset A of $\mathrm{Seq}\,\kappa$ we shall write $\min A$ to denote the set of all minimal elements of A (with respect to the tree ordering on $\mathrm{Seq}\,\kappa$, of course).

If F is a filtration on $\mathrm{Seq}\,\kappa$ and $v \in \mathrm{Seq}\,\kappa$ we shall denote by F_v the derived filtration on $\mathrm{Seq}\,\kappa$ defined by the formula $F_v(s) = F(v^\frown s)$.

Assume now that S and $\{T_v : v \in \mathrm{Seq}\,\kappa\}$ are subtrees of $\mathrm{Seq}\,\kappa$. We then define their "sum" by

$$S \oplus \{T_v : v \in \mathrm{Seq}\,\kappa\} = S \cup \{v^\frown t : v \in \min(\mathrm{Seq}\,\kappa \setminus S) \wedge t \in T_v\}.$$

Obviously, this sum is again a subtree of $\mathrm{Seq}\,\kappa$.

If moreover f and $g = \{g_v : v \in \mathrm{Seq}\,\kappa\}$ are functions with $\mathrm{dom}\, f = S$ and $\mathrm{dom}\, g_v = T_v$ then we define $f \oplus \{g_v : v \in \mathrm{Seq}\,\kappa\} = f \oplus g$ by putting

$$\mathrm{dom}(f \oplus g) = S \oplus \{T_v : v \in \mathrm{Seq}\,\kappa\}$$

and

$$(f \oplus g)(x) = \begin{cases} f(x) & \text{for } x \in S \\ g_v(t) & \text{for } x = v^\frown t \text{ with } v \in \min(\mathrm{Seq}\,\kappa \setminus S),\ t \in T. \end{cases}$$

A subtree of $\mathrm{Seq}\,\kappa$ is called **well-founded** if and only if it does not possess any infinite branches. Note that if S and $\{T_v : v \in \mathrm{Seq}\,\kappa\}$ are all well-founded then so is $S \oplus \{T_v : v \in \mathrm{Seq}\,\kappa\}$.

Now let $0 < \lambda \le \kappa$ be cardinals and F be a filtration on $\mathrm{Seq}\,\kappa$. We say that a function f is **λ-good for F** if and only if $\mathrm{dom}\, f$ is a well-founded subtree of

$\operatorname{Seq}\kappa$, moreover $f[V] = \lambda$ whenever V is open in $X(F)$ with $\emptyset \in V$. As an easy (but useful) illustration of this concept we present the following result.

LEMMA 3.10: *For each $0 < n < \omega$ and for any filtration F on κ there is a function f which is n-good for F.*

Proof. Let $\operatorname{dom} f = \{s \in \operatorname{Seq}\kappa : |s| < n\}$ and $f(s) = |s|$. ∎

The next result shows the relevance of these concepts to resolvability.

THEOREM 3.11: *Let F be an filtration on $\operatorname{Seq}\kappa$. If there are λ-good functions f_s for F_s for all $s \in \operatorname{Seq}\kappa$ then $X(F)$ is λ-resolvable.*

Proof. Define the sequence of functions $g_0, g_1, \dots$ by recursion as follows: $g_0 = f_\emptyset$ and $g_{n+1} = g_n \oplus \{f_s : s \in \operatorname{Seq}\kappa\}$ for $n < \omega$. It is easy to check that then $g_\omega = \bigcup_{n<\omega} g_n$ maps $\operatorname{Seq}\kappa$ to λ, i.e. $\operatorname{dom} g_\omega = \operatorname{Seq}\kappa$. Indeed, if $s \in \operatorname{Seq}\kappa$ with $|s| = n$ then there is a $k \le n$ with $s \in \operatorname{dom} g_k$.

We show next that $g_\omega[V] = \lambda$ holds for any nonempty open set V in $X(F)$. Let n be such that $V \cap \operatorname{dom} g_n \ne \emptyset$ and pick $v \in V \cap \operatorname{dom} g_n$. Clearly, there is an extension s of v with $s \in V \cap \min(\operatorname{Seq}\kappa \setminus \operatorname{dom} g_n)$. Now let

$$W = \{t \in \operatorname{Seq}\kappa : s^\frown t \in V\}$$

then $\emptyset \in W$ and W is open in $X(F_s)$, hence $f_s[W] = \lambda$ because f_s is λ-good for F_s. But we clearly have $g_\omega(s^\frown t) = f_s(t)$ for all $t \in \operatorname{dom} f_s$, hence we have $g_\omega[V] = \lambda$ as well.

But then $\{g_\omega^{-1}(\alpha) : \alpha < \lambda\}$ is a pairwise disjoint family of dense sets in $X(F)$. ∎

The following stepping-up type result will turn out to be very useful.

LEMMA 3.12: *Assume that F is a filtration on $\operatorname{Seq}\kappa$ such that $F(\emptyset)$ is λ-descendingly incomplete, moreover, for every cardinal $\mu < \lambda$ and every ordinal $\alpha < \kappa$ there is a μ-good function f^α_μ for $F_{\langle\alpha\rangle}$. Then there is a λ-good function f for F.*

Proof. Fix a continuously decreasing sequence $\{X_\xi : \xi < \lambda\} \subset F(\emptyset)$ with empty intersection. For any ordinal $\nu < \lambda$ let us put $I_\nu = X_\nu \setminus X_{\nu+1}$, then we clearly have $\kappa = \bigcup\{I_\nu : \nu < \lambda\}$. For each $0 < \nu < \lambda$ fix a map $h_\nu : |\nu| \overset{onto}{\to} \nu$.

We now define the desired map f with the following stipulations:

$$\operatorname{dom} f = \{\emptyset\} \cup \bigcup_{\nu<\lambda} \{\langle\alpha\rangle^\frown t : \alpha \in I_\nu \text{ and } t \in \operatorname{dom} f^\alpha_{|\nu|}\} ,$$

and for $s \in \operatorname{dom} f$

$$f(s) = \begin{cases} 0 & \text{if } s = \emptyset\,, \\ h_\nu(f^\alpha_{|\nu|}(t)) & \text{if } s = \langle\alpha\rangle^\frown t \text{ with } \alpha \in I_\nu,\ t \in \operatorname{dom} f^\alpha_{|\nu|}\ . \end{cases}$$

Clearly, f is well-defined and $\operatorname{dom} f$ is well-founded. If V is open in $X(F)$ with $\emptyset \in V$ then we have $V \cap S_\emptyset \in F(\emptyset)$ and hence

$$\sup\{\nu : \exists\alpha \in I_\nu \text{ with } \langle\alpha\rangle \in V\} = \lambda.$$

But $\langle\alpha\rangle \in V$ and $\alpha \in I_\nu$ imply $f^\alpha_{|\nu|}[\{s : \langle\alpha\rangle^\frown s \in V\}] = |\nu|$ and so $f[V] \supset \nu$, hence we have $f[V] = \lambda$. ∎

THEOREM 3.13: *Let F be a filtration on $\operatorname{Seq}\kappa$ and λ be an infinite cardinal such that $F(t)$ is μ-descendingly incomplete whenever $t \in \operatorname{Seq}\kappa$ and $\omega \le \mu \le \lambda$. Then there are λ-good functions for all the derived filtrations F_s and hence $X(F)$ is λ-resolvable.*

Proof. The proof goes by a straight-forward induction on λ, using Lemma 3.12 and the fact that our assumption on F is automatically valid also for all the derived filtrations F_s. The starting case $\lambda = \omega$ also uses Lemma 3.10. The last statement is immediate from Theorem 3.11. ∎

A uniform ultrafilter on κ is trivially κ-descendingly incomplete. So if $\kappa = \omega_n < \aleph_\omega$, then it follows by n repeated applications of the above mentioned result of Kunen and Prikry that any uniform ultrafilter on κ is μ-descendingly incomplete for all μ with $\omega \le \mu \le \kappa$. Thus we get from Theorem 3.13 the following result.

COROLLARY 3.14: *Assume that $\kappa < \aleph_\omega$ and F is any uniform ultrafiltration on $\operatorname{Seq}\kappa$ (i.e. $F(t)$ is uniform for all $t \in \operatorname{Seq}\kappa$). Then the space $X(F)$ is κ-resolvable.*

We now recall a definition from [9], see also [12].

Definition 3.15: Let X be a space and μ be an infinite cardinal number. We say that $x \in X$ is a T_μ point of X if for every set $A \in [X]^{<\mu}$ there is some $B \in [X\setminus A]^{<\mu}$ such that $x \in \overline{B}$. We shall use $T_\mu(X)$ to denote the set of all T_μ points of X.

The following result is an easy consequence of Lemma 1.3 from [9]. In the particular case when μ is a successor cardinal it follows from Proposition 2.1 of [12].

LEMMA 3.16: *If $|X| = \mu$ is a regular cardinal and $T_\mu(X)$ is dense in X then X is μ-resolvable.*

This result will enable us to transfer certain results from spaces of the form $X(F)$, where F is a uniform ultrafiltration on $\operatorname{Seq}\kappa$ for some regular cardinal κ, to monotonically normal and even more general spaces.

Let us recall from Section 1 that every monotonically normal space is SD. In fact, as monotone normality is a hereditary property, it is even hereditarily SD (in short: HSD). We shall need below a property that is strictly between SD and HSD, namely that all **dense** subspaces are SD, we shall denote this property by DSD. It can be shown that the Čech-Stone remainder ω^*, for instance, is DSD but not HSD.

THEOREM 3.17: *Assume that $\kappa = \operatorname{cf}(\kappa) \geq \lambda$. Then the following are equivalent.*

(1) *If X is a DSD space with $|X| = \Delta(X) = \kappa$ then X is λ-resolvable.*
(2) *If X is a MN space with $|X| = \Delta(X) = \kappa$ then X is λ-resolvable.*
(3) *For every uniform ultrafiltration F on $\operatorname{Seq}\kappa$ the space $X(F)$ is λ-resolvable.*

Proof. Of course, only (3) $\Rightarrow$ (1) requires proof. So assume (3) and consider a DSD space X with $|X| = \Delta(X) = \kappa$. If $T_\kappa(X)$ is dense in X then, by Lemma 3.16 X is even κ-resolvable and we are done.

Otherwise, in view of Lemma 1.1, we may assume that actually $T_\kappa(X) = \emptyset$. In this case, for every point $x \in X$ there is a set $A_x \in [X]^{<\kappa}$ such that $x \in A_x$ and for $D_x = X \setminus A_x$ no $B \in [D_x]^{<\kappa}$ has x in its closure. Note that by $\Delta(X) = \kappa$ each D_x is dense in X.

But X is DSD, hence for every x there is a strongly discrete set $S_x \subset D_x$ with $x \in \overline{S_x}$. (Note that $S \subset D_x$ is strongly discrete in D_x if and only if it is so in X for D_x is dense.)

Next, by recursion on $|t|$, we define points x_t and open sets U_t in X as follows. First pick any point $x_\emptyset \in X = U_\emptyset$. If $x_t \in U_t$ has been defined then fix a one-to-one enumeration of $S_{x_t} \cap U_t = \{x_{t^\frown\alpha} : \alpha < \kappa\}$ and choose $\{U_{t^\frown\alpha} : \alpha < \kappa\}$ to be pairwise disjoint open neighbourhoods of them, all contained in U_t. Clearly, then the map $h : \operatorname{Seq}\kappa \to X$ that maps t to $h(t) = x_t$ is injective.

Next, for any $t \in \mathrm{Seq}\,\kappa$ extend the trace of the neighbourhood filter of x_t on $S_{x_t} \cap U_t$ to an ultrafilter u_t and define $F(t) = h^{-1}[u_t]$, which is an ultrafilter on $S_t = \{t^\frown\alpha : \alpha < \kappa\}$. It follows from our assumptions that every $F(t)$ is uniform and therefore $X(F)$ is λ-resolvable. But the subspace topology on $h[\mathrm{Seq}\,\kappa]$ in X is clearly coarser than the h-image of τ_F, hence it is also λ-resolvable. By Lemma 1.1, this completes our proof. ∎

Corollary 3.18: *Let X be any DSD space of cardinality $< \aleph_\omega$. Then X is maximally resolvable. In particular, all MN spaces of size $< \aleph_\omega$ are maximally resolvable.*

Proof. Clearly, every open set U in X includes another open set V such that $|V| = \Delta(V)$. But every open subspace of a DSD space is again DSD, so Theorem 3.17 and Corollary 3.14 imply that V is $|V|$-resolvable. But $\Delta(X) \leq |V|$, hence each such V is $\Delta(X)$-resolvable and so, in view of Lemma 1.1, X is maximally resolvable. ∎

We conclude by listing a few open problems that we find especially interesting.

Problem 3.19: (1) Is there a ZFC example of a monotonically normal space that is not maximally resolvable?

(2) Is it consistent to have a monotonically normal space X of cardinality less than the first measurable such that $\Delta(X) > \omega$ but X is not ω_1-resolvable?

(3) Is every crowded monotonically normal space almost $\mathfrak{c}$-resolvable?

Concerning problem (3) we have the following (very) partial result: Every **countable** crowded DSD space is almost $\mathfrak{c}$-resolvable.

References

[1] S. Ben-David and M. Magidor, *The weak $\square^*$ is really weaker than the full $\square$*, Journal of Symbolic Logic **51** (1986), 1029–1033.

[2] J. Ceder and T. Pearson, *On products of maximally resolvable spaces*, Pacific Journal of Mathematics **22** (1967), 31–45.

[3] A. Dow, M. G. Tkachenko, V. V. Tkachuk, and R. G. Wilson,*Topologies generated by discrete subspaces*, Glasnik Matematički. Serija III **37**(57) (2002), 187–210.

[4] F. W. Eckertson, *Resolvable, not maximally resolvable spaces*, Topology and its Applications **79** (1997), 1–11.

[5] A. G. El'kin, *Resolvable spaces which are not maximally resolvable*, Vestnik Moskovskogo Universiteta. Seriya I. Matematika. Mekhanika **24** (1969), 66–70.

[6] L. Hegedűs, *Szűrők és Lyukak*, Master's Thesis, in Hungarian, Eötvös Loránd University, Budapest, 1998.

[7] E. Hewitt, *A problem of set theoretic topology*, Duke Mathematical Journal **10** (1943), 309–333.

[8] I. Juhász, L. Soukup, and Z. Szentmiklóssy, *$\mathcal{D}$-forced spaces: a new approach to resolvability*, Topology and its Applications **153** (2006), 1800–1824.

[9] I. Juhász, L. Soukup, and Z. Szentmiklóssy, *Resolvability of spaces having small spread or extent*, Topology and its Applications **154** (2007), 144–154.

[10] K. Kunen and K. Prikry, *On descendingly incomplete ultrafilters*, Journal of Symbolic Logic **36** (1971), 650–652.

[11] M. Magidor, private communication.

[12] O. Pavlov, *On resolvability of topological spaces*, Topology and its Applications **126** (2002), 37–47.

[13] S. W. Williams and H. X. Zhou, *Strong versions of normality*, in *General topology and applications (Staten Island, NY, 1989)*, Lecture Notes in Pure and Applied Mathematics, 134, Dekker, New York, 1991, pp. 379–389.